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STATISTICAL MECHANICS OF GASEOUS SUSPENSIONS. DYNAMIC AND SPECTRAL EQUATIONS

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We present the model of random, pseudoturbulent phase motions in concentrated, disperse, gas-suspended particles systems, based on the development of ideas expounded in our previous papers [1 – 3] and [4]. This model enables us, in principle, to construct a structural theory of gaseous suspensions when the flow is pseudoturbulent [3], to compute the corresponding transfer coefficients and to formulate the dynamic equations of motion.

Papers [1 – 3] considered pulsating motions of a two-phase disperse system using the statistical approach and developed a general method of the quantitative treatment of the pulsations and their influence on the average motion of the system. At the same time, ways were indicated towards constructing a non-Newtonian mechanics of disperse systems.

The model [3] however, retains a number of unsolved difficulties. First of them concerns the fact that the proposed model is based on the use of certain random forces acting on the phases in random motion, and of the statistical white noise, the latter allowing the description of not only of the orderly degeneration of the fluctuations of the averaged hydrodynamic field of a disperse system, but also of their random accumulation. The forces and the white noise enter [3] separately, although the general physical considerations imply that a mere appearance of the white noise should be the result of the action of the random forces.

Secondly, positive results are obtained under the additional assumptions of the type of the hypothesis of the statistical dependence of the positions of the separate particles in space. This hypothesis is valid, strictly speaking, only, when the volume of the disperse system over which the pulsations are analized, is sufficiently large.

Thirdly, linearization of the stochastic equations requires an assumption of smallness of the random perturbations of the average motion, and this, in fact, is not always true. Finally, when the dynamic equations are formulated, an ambiguity arises connected with the notation used to write the terms describing the pulsatory stresses. The law of conservation of the moment of momentum implies well defined relations, which should be satisfied by the coefficients of the pulsating phase viscosities obtained by completely independent methods.

In the theory proposed we use a method which is basically similar to that used to describe the turbulence of a single phase medium [4]; the shortcomings listed above are eliminated.

1. Dynamic and stochastic equations. Below we consider a system of particles suspended in a gas, neglecting for simplicity the impulse in the gas and the viscous dissipation of energy taking place in it. Using the concept of a continuous medium as an approximation we can find, as in [3], the velocities of the gaseous and dispersed phase by averaging over the volumes containing $N \gg 1$ particles. If L is the linear dimension of such a volume, then we obtain the following relations for the velocities of the phases, the gas pressure and the volume concentration of the disperse system

$$\mathbf{v}_L = \mathbf{v} + \mathbf{v}_L', \quad \mathbf{w}_L = \mathbf{w} + \mathbf{w}_L', \quad p_L = d_2 \pi_L = d_2 (\pi + \pi_L'), \quad \rho_L = \rho + \rho_L' \tag{1.1}$$

respectively, where the first terms in the right-hand sides are obtained by the averaging over the whole ensemble (i.e. formally as N and $L \to \infty$ so that $\mathbf{v}_L \to \mathbf{v}$), and the second terms represent the random pulsations of the corresponding magnitudes. Here and below π denotes the pressure divided by the density d_2 of the material forming the particles.

When subjected to such an averaging process, the pulsations of dimensions smaller than L, partly disappear. We shall describe their influence on the averaged motion and large amplitude pulsations using, similarly to [4], the pulsation pressure tensor and the pseudoturbulent viscosity. These quantities are analogous to the ordinary pressure and viscosity which depend on the molecular motions (*). Remembering the necessity of satisfying the law of conservation of the angular momentum, we shall symmetrize the pseudoturbulent viscous stress tensor [4] at once, obtaining the following formal expressions for the tensor $\mathbf{P}_L$ of the pulsatory (pseudoturbulent) (**) pressure of the dispersed phase and for the pseudoturbulent stress tensor $\boldsymbol{\tau}_L$ in this phase:

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*) We note that the pulsatory pressure did not appear in [4], where only homogenous turbulence was considered for the reason that the derivatives of the average characteristics of the homogeneous turbulent field with respect to the coordinates are equal to zero. This implies that this pressure had no influence on the motion.

**) Here we use the terminology of [3]. Pseudoturbulent motions, unlike the usual turbulence, are mainly supported by the action of the gravity and of the viscous phase interactions, on the concentration fluctuations of the system (also see below).

$$\mathbf{P}_L = \|P_{L,ij}\|, \quad \boldsymbol{\tau}_L = \|\tau_{L,ij}\|, \quad P_{L,ij} = \rho\, d_2 \Pi_{L,ij}$$
$$\tau_{L,ij} = \eta_{L,ik}\frac{\partial w_{Lj}}{\partial x_k} + \eta_{L,jk}\frac{\partial w_{Li}}{\partial x_k}, \qquad \eta_L = \rho\, d_2 \nu_L, \; \nu_L = \|\nu_{L,ij}\| \tag{1.2}$$

Here η_L and ν_L denote the tensors of the dynamic and pseudoturbulent viscosity of the dispersed phase, while Π_L is the tensor of the RMS pulsatory velocities of this phase, and they all depend on the small (compared with the average dimension L) random motions. Below we shall give a more detailed definition of those tensors, now it will suffice to say that all the tensors appearing in (1.2) are assumed to be, within some approximation, dependent on the conditions of the average motion and on the physical phase parameters only. This essentially corresponds to the analogous assumptions used in the kinetic theory of gas. Relations of the type (1.2) can also be set up for the dispersing phase, but, when we neglect the impulse and molecular viscosity of the gas, we can also neglect all the pseudoturbulent motions of the gas and their influence on the averaged motion of the two-phase system.

Equations of the conservation of mass of the phases have the usual form

$$\frac{\partial \rho_L}{\partial t} - \frac{\partial\,[(1-\rho_L)\mathbf{v}_L]}{\partial \mathbf{r}} = 0, \qquad \frac{\partial \rho_L}{\partial t} + \frac{\partial\,(\rho_L \mathbf{w}_L)}{\partial \mathbf{r}} = 0 \tag{1.3}$$

Equations of the conservation of impulse of the phases with (1.2) taken into account, can be written as

$$0 = -(1-\rho_L)\frac{\partial \pi_L}{\partial \mathbf{r}} - \beta\rho_L K_L(\mathbf{v}_L - \mathbf{w}_L), \qquad \beta = \frac{9}{2}\frac{\varkappa \nu_0}{a^2}, \qquad \varkappa = \frac{d_1}{d_2}$$
$$\rho_L\left(\frac{\partial}{\partial t} + \mathbf{w}_L\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{w}_L = -\rho_L\frac{\partial \pi_L}{\partial \mathbf{r}} + \rho_L \mathbf{g} + \beta\rho_L K_L(\mathbf{v}_L - \mathbf{w}_L) - \frac{\partial\,(\rho \Pi_L)}{\partial \mathbf{r}} + \tag{1.4}$$
$$+ \frac{\partial\,(\rho\sigma_L)}{\partial \mathbf{r}}, \quad \sigma_L = \frac{\tau_L}{\rho\, d_2}, \quad \sigma_{L,ij} = \nu_{L,ik}\frac{\partial w_{Lj}}{\partial x_k} + \nu_{L,jk}\frac{\partial w_{Li}}{\partial x_k}, \quad K_L = K(\rho_L)$$

where d_1 and $\mu_0 = d_1\nu_0$ denote the density and viscosity of the gas, g is the free-fall acceleration vector, $K = K(\rho)$ is the function expressing the influence of the imposed constraints on the Stokes mode of flow past the particles of radius a. Expression

$$\mathbf{f}_L = -\rho_L d_2 \nabla\pi + \beta\, d_2 \rho_L K_L(\mathbf{v}_L - \mathbf{w}_L)$$

describes the force of interaction between the phases, referred to the unit volume of the mixture and valid for small d_1 and a (1.4).

Inserting (1.1) into (1.3) and averaging over the whole ensemble, we obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\rho - (1-\rho)\frac{\partial \mathbf{v}}{\partial \mathbf{r}} = \langle R_L^{(g)}\rangle, \qquad R_L^{(g)} = -\frac{\partial\,(\rho_L'\mathbf{v}_L')}{\partial \mathbf{r}}$$
$$\left(\frac{\partial}{\partial t} + \mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)\rho + \rho\frac{\partial \mathbf{w}}{\partial \mathbf{r}} = \langle R_L^{(p)}\rangle, \qquad R_L^{(p)} = -\frac{\partial\,(\rho_L'\mathbf{w}_L')}{\partial \mathbf{r}} \tag{1.5}$$

Similarly, limiting ourselves to the second order terms in the pseudoturbulent pulsations and utilizing the relation $\rho_L\left(\mathbf{w}_L\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{w}_L \Rightarrow \frac{\partial}{\partial \mathbf{r}}(\rho_L \mathbf{w}_L * \mathbf{w}_L) + \mathbf{w}_L\frac{\partial \rho_L}{\partial t}$ obtained from the second Eq. of (1.3), the asterisk denoting the dyadic product, we obtain from (1.4) the following equations for the averaged motion:

$$0 = -(1-\rho)\frac{\partial \pi}{\partial \mathbf{r}} - \beta\rho K(\mathbf{v} - \mathbf{w}) + \langle S_L^{(g)}\rangle + \langle S_L^{(i)}\rangle$$

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{w}=-\rho\frac{\partial \pi}{\partial \mathbf{r}}+\rho\mathbf{g}+\beta\rho K(\mathbf{v}-\mathbf{w})-\frac{\partial(\rho\Pi_L)}{\partial \mathbf{r}}+\frac{\partial(\rho\sigma_L{}^\circ)}{\partial \mathbf{r}}+$$

$$+\langle \mathbf{S}_L^{(p)}\rangle-\langle \mathbf{S}_L^{(i)}\rangle \qquad \mathbf{S}_L^{(g)}=0, \quad \mathbf{S}_L^{(i)}=\rho_L{}'\frac{\partial \pi_L{}'}{\partial \mathbf{r}}-\beta\left[\frac{d(\rho K)}{d\rho}\rho_L{}'(\mathbf{v}_L{}'-\mathbf{w}_L{}')+\right.$$

$$\left.+\frac{1}{2}\frac{d^2(\rho K)}{d\rho^2}(\mathbf{v}-\mathbf{w})\rho_L{}'^2\right] \qquad (1.6)$$

$$\mathbf{S}_L^{(p)}=-\frac{\partial(\rho_L{}'\mathbf{w}_L{}')}{\partial t}-\rho\frac{\partial(\mathbf{w}_L{}'*\mathbf{w}_L{}')}{\partial \mathbf{r}}-\left(\mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)\rho_L{}'\mathbf{w}_L{}'$$

where $\mathbf{w}$ in σ°_L replaces $\mathbf{w}_L$ in σ_L in (1.4). The last terms in (1.5) and (1.6) describe the influence of large scale pulsations on the averaged motion. In the case of a random motion of an incompressible, single phase medium, they reduce to the Reynolds stresses governed by the large scale pulsations of this medium [4].

For the pseudoturbulent fluctuations, (1.3) and (1.5) together with (1.4) and (1.6) yield the following stochastic equations:

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\rho_L{}'-(1-\rho)\frac{\partial \mathbf{v}_L{}'}{\partial \mathbf{r}}+\mathbf{v}_L{}'\frac{\partial \rho}{\partial \mathbf{r}}+\rho_L{}'\frac{\partial \mathbf{v}}{\partial \mathbf{r}}=Q_L^{(g)}=R_L^{(g)}-\langle R_L^{(g)}\rangle$$

$$\left(\frac{\partial}{\partial t}+\mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)\rho_L{}'+\rho\frac{\partial \mathbf{w}_L{}'}{\partial \mathbf{r}}+\mathbf{w}_L{}'\frac{\partial \rho}{\partial \mathbf{r}}+\rho_L{}'\frac{\partial \mathbf{w}}{\partial \mathbf{r}}=Q_L^{(p)}=R_L^{(p)}-\langle R_L^{(p)}\rangle$$

$$-(1-\rho)\frac{\partial \pi_L{}'}{\partial \mathbf{r}}-\left[\beta\frac{d(\rho K)}{d\rho}(\mathbf{v}-\mathbf{w})-\frac{\partial \pi}{\partial \mathbf{r}}\right]\rho_L{}'-\beta\rho K(\mathbf{v}_L{}'-\mathbf{w}_L{}')=\mathbf{F}_L^{(i)}$$

$$(1.7)$$

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{w}_L{}'+\rho\left(\mathbf{w}_L{}'\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{w}=-\rho\frac{\partial \pi_L{}'}{\partial \mathbf{r}}+\left[-\left(\frac{\partial}{\partial t}+\mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{w}-\right.$$

$$\left.-\frac{\partial \pi}{\partial \mathbf{r}}+\mathbf{g}+\beta\frac{d(\rho K)}{d\rho}(\mathbf{v}-\mathbf{w})\right]\rho_L{}'+\beta\rho K(\mathbf{v}_L{}'-\mathbf{w}_L{}')+\frac{\partial(\rho\sigma_L{}')}{\partial \mathbf{r}}-\mathbf{F}_L^{(p)}+\mathbf{F}_L^{(i)},$$

$$\mathbf{F}_L^{(i)}=-\mathbf{S}_L^{(i)}+\langle \mathbf{S}_L^{(i)}\rangle, \qquad \mathbf{F}_L^{(p)}=-\mathbf{S}_L^{(p)}+\langle \mathbf{S}_L^{(p)}\rangle$$

where $\sigma_L{}'$ differs from σ_L, replacing $\mathbf{w}_L$ by $\mathbf{w}_L{}'$.

In accordance with the model given in [3] we assume that the amplitues of the pseudoturbulence are much smaller than the dimensions of the averaged motion. Passing in (1.5) and (1.6) to the limit as $L\to\infty$ and taking into account that R_L and S_L tend, at the same time, to zero, we obtain the dynamic equations of motion in the following form:

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\rho-(1-\rho)\frac{\partial \mathbf{v}}{\partial \mathbf{r}}=0, \qquad \left(\frac{\partial}{\partial t}+\mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)\rho+\rho\frac{\partial \mathbf{w}}{\partial \mathbf{r}}=0$$

$$-(1-\rho)\frac{\partial \pi}{\partial \mathbf{r}}-\beta\rho K(\mathbf{v}-\mathbf{w})=0, \qquad \rho\left(\frac{\partial}{\partial t}+\mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{w}=$$

$$=-\rho\frac{\partial \pi}{\partial \mathbf{r}}+\rho\mathbf{g}+\beta\rho K(\mathbf{v}-\mathbf{w})-\frac{\partial(\rho\Pi)}{\partial \mathbf{r}}+\frac{\partial(\rho\sigma^\circ)}{\partial \mathbf{r}} \qquad (1.8)$$

Here the tensors Π and σ° denote the limit values of Π_L and $\sigma_L{}^\circ$ as $L\to\infty$ and they describe the influence of all pseudoturbulent motions on the averaged motion of the disperse system. Comparing (1.8) with (1.5) and (1.6), we obtain

$$\langle \mathbf{S}_L^{(i)}\rangle=\langle R_L^{(g)}\rangle=\langle R_L^{(p)}\rangle=0, \qquad \langle \mathbf{S}_L^{(p)}\rangle\neq 0$$

Relations (1.8) also yield the equation of the conservation of impulse of the dispersoid.

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{w}\,\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{w}=-\,\frac{\partial \pi}{\partial \mathbf{r}}+\rho\mathbf{g}-\frac{\partial(\rho\Pi)}{\partial \mathbf{r}}+\frac{\partial(\rho\sigma^\circ)}{\partial \mathbf{r}} \qquad (1.9)$$

The assumption discussed in [3] that the significant variations in the values of parameters of the averaged motion are much larger than the amplitudes of the pseudoturbulence, enables us to simplify the stochastic equations (1.7). Representing all random magnitudes as stochastic Fourier-Stieltjes integrals with random measures we obtain, in the local coordinate system where $\mathbf{w}=0$, $\mathbf{v}=\mathbf{u}$ the following Eqs. :

$$(\omega+\mathbf{uk})\,dZ_\rho-(1-\rho)\mathbf{k}\,d\mathbf{Z}_v=dZ_Q^{(g)},\qquad \omega\,dZ_\rho+\rho\mathbf{k}\,d\mathbf{Z}_w=dZ_Q^{(p)}$$

$$-\,i\,(1-\rho)\,\mathbf{k}\,dZ_\pi-\left(\beta\,\frac{d\,(\rho K)}{d\rho}\,\mathbf{u}-\frac{\partial\pi}{\partial\mathbf{r}}\right)dZ_\rho-\beta\rho K\,(d\mathbf{Z}_v-d\mathbf{Z}_w)=d\mathbf{Z}_F^{(i)}$$

$$(1.10)$$

$$-\,\rho\,(i\omega+\nu(k)\,\mathbf{kk})\,d\mathbf{Z}_w-i\rho\mathbf{k}\,dZ_\pi+\left(-\frac{\partial\mathbf{w}}{\partial t}-\frac{\partial\pi}{\partial\mathbf{r}}+\mathbf{g}+\beta\,\frac{d\,(\rho K)}{d\rho}\,\mathbf{u}\right)dZ_\rho+$$

$$+\,\beta\rho K\,(d\mathbf{Z}_v-d\mathbf{Z}_w)-\rho\,(\nu(k)\,\mathbf{k})\,(\mathbf{k}\,d\mathbf{Z}_w)=-\,d\mathbf{Z}_F^{(i)}+d\mathbf{Z}_F^{(p)}$$

where ω is the frequency, $\mathbf{k}$ is the wave vector and dZ_Q denote the differentials of the random measures of the processes Q multiplied by $(-\,i)$.

Introduction of the Fourier-Stieltjes integrals enables us to sharpen definitions of the pulsations of the type φ'_L, as well as of Π_L and ν_L. We have

$$\mathbf{w}_L'\approx\int\limits_{\omega}\int\limits_{k^\circ<k}e^{i(\omega t+\mathbf{k}^\circ\mathbf{r})}\,d\mathbf{Z}_w,\qquad \Pi_L\approx\int\limits_{\omega}\int\limits_{k'>k}\mathrm{Re}\,\langle d\mathbf{Z}_w{}^{*}*d\mathbf{Z}_w\rangle$$

Moreover, when deriving Eqs. (1.10), we assumed the following expression for the pseudoturbulent transfer of impulse between the fluctuations whose wave numbers are $k^\circ<k$ and the vortices whose wave numbers are $k'>k$

$$\left(\nu_L\,\frac{\partial}{\partial\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{r}}\right)\mathbf{w}_L'\approx-\int\limits_{\omega}\int\limits_{k^\circ<k}\nu(k^\circ)\,\mathbf{k}^\circ\mathbf{k}^\circ e^{i(\omega t+\mathbf{k}^\circ\mathbf{r})}\,d\mathbf{Z}_w$$

$$\nu(k)=\alpha\int\limits_{\omega}\int\limits_{k'>k}\int\limits_{0}^{\infty}\mathrm{Re}\,\{e^{i\omega\tau}\langle d\mathbf{Z}_w{}^{*}*d\mathbf{Z}_w\rangle\}\,d\tau$$

In the above expressions $k\sim L^{-1}$ and α is taken as a semi-empirical constant of the order of unity. The last expressions correspond to the model, according to which the vortices with $k'>k^\circ$ influence the random motion with the wave number k° (of course $k^\circ\leqslant k$) through the mechanism of the pseudoturbulent viscosity, while the vortices with $k'<k^\circ$ – through random forces and the divergence of flows in the right hand sides (1.10). The conceptual similarity to the Heisenberg's hypothesis [5] is obvious (more detailed discussion will appear at the end of Section 2).

Homogeneous equations (1.7) or (1.10) describe an ordered degeneration of some initial fluctuation field $\mathbf{v}_L'(t_0)$, $\mathbf{w}_L'(t_0)$, $\pi_L'(t_0)$ and $\rho_L'(t_0)$ caused by the pressure forces, viscosity, e.a. Divergencies of random flows Q and forces $\mathbf{F}$ describe the accumulation of new fluctuations. We can separate these processes by introducing, as in [6], two methods of averaging, the complete one used above, and the averaging when the initial state is fixed. We can easily see that the time scale of the first process is comparable with the time of decay of the correlation functions of the pseudoturbulent field and, that it considerably exceeds the time required to produce a significant variation in the random terms appearing in the right-hand sides of (1.7) (see also [4]). Therefore, when investigating the dynamics of the fluctuation degeneracy we can consider, in the first approximation, Q and $\mathbf{F}$ in (1.7) as being localized in time and assume the spectral densities of their correlation functions to be independent of the frequency ω (assumptions used in [6] that the random forces are spatially localized, is not employed here).

Following the terminology of [6] we can call the time scales discussed above, the external and internal temporal pseudoturbulence scales; they have the same sense as the external and internal spatial scales of the turbulence in a single phase fluid.

2. Spectral equations. Complete definition of the random processes introduced in Section 1 and of the dynamic equations (1.8) demands the knowledge of the statistical characteristics of the random processes which appear in the right-hand sides of (1.7) or (1.10). Below we derive the spectral equations for these characteristics.

Eliminating $d\mathbf{Z}_v$ and dZ_π from (1.10) we obtain

$$\omega\, dZ_\rho + \rho \mathbf{k}\, d\mathbf{Z}_w = dZ_Q^{(p)}, \qquad -\rho\,(i\omega + \nu(k)\,\mathbf{kk})\, d\mathbf{Z}_w +$$

$$+ (\mathbf{B}\omega + \mathbf{C})\, dZ_\rho = d\mathbf{Z}_F^{(p)} + \frac{1}{1-\rho}\left(\frac{dZ_Q^{(g)}}{1-\rho} + \frac{dZ_Q^{(p)}}{\rho}\right)\frac{\mathbf{k}}{k^2} + \nu(k)\,\mathbf{k}\, dZ_Q^{(p)}$$

$$\mathbf{B} = \nu(k)\,\mathbf{k} + \frac{\beta K}{(1-\rho)^2}\frac{\mathbf{k}}{k^2}, \qquad \mathbf{C} = -\frac{\partial \mathbf{w}}{\partial t} + \mathbf{g} + \frac{(\mathbf{A}\mathbf{k})\,\mathbf{k}}{k^2} \tag{2.1}$$

$$\mathbf{A} = \frac{1}{1-\rho}\left(\beta\, \frac{d(\rho K)}{d\rho}\mathbf{u} + \frac{\rho}{1-\rho}\beta K \mathbf{u} - \frac{\partial \pi}{\partial \mathbf{r}}\right)$$

The quantities $d\mathbf{Z}_v$ and dZ_π satisfy

$$\beta\rho K\,(d\mathbf{Z}_v - d\mathbf{Z}_w) - (\mathbf{B}'\omega + \mathbf{C}')\, dZ_\rho = -\, d\mathbf{Z}_F^{(i)} - \left(\frac{dZ_Q^{(g)}}{1-\rho} + \frac{dZ_Q^{(p)}}{\rho}\right)\frac{\mathbf{k}}{k^2}$$

$$-\, i\, dZ_\pi = \frac{(\mathbf{A}\mathbf{k})}{k^2}\, dZ_\rho + \frac{\beta K \omega}{(1-\rho)^2 k^2}\, dZ_\rho - \frac{1}{1-\rho}\left(\frac{dZ_Q^{(g)}}{1-\rho} + \frac{dZ_Q^{(p)}}{\rho}\right)\frac{1}{k^2} \tag{2.2}$$

$$\mathbf{B}' = \frac{\beta K}{1-\rho}\frac{\mathbf{k}}{k^2}, \qquad \mathbf{C}' = -\beta\, \frac{d(\rho K)}{d\rho}\mathbf{u} + \frac{\partial \pi}{\partial \mathbf{r}} + (1-\rho)\frac{(\mathbf{A}\mathbf{k})\,\mathbf{k}}{k^2}$$

At this point we shall introduce further random processes

$$\rho\, dZ_1 = dZ_Q^{(p)'}, \qquad \rho\, d\mathbf{Z}_2 = d\mathbf{Z}_F^{(p)} + \frac{1}{1-\rho}\left(\frac{dZ_Q^{(g)}}{1-\rho} + \frac{dZ_Q^{(p)}}{\rho}\right)\frac{\mathbf{k}}{k^2} +$$

$$+ \nu(k)\,\mathbf{k}\, dZ_Q^{(p)}, \qquad \rho\, d\mathbf{Z}_3 = d\mathbf{Z}_F^{(i)} + \left(\frac{dZ_Q^{(g)}}{1-\rho} + \frac{dZ_Q^{(p)}}{\rho}\right)\frac{\mathbf{k}}{k^2}$$

From Eqs. (2.1) we obtain

$$dZ_\rho = \rho\,(i\omega^2 + 2b\omega + c)^{-1}\,[\mathbf{k}\, d\mathbf{Z}_2 + (i\omega + \nu(k)\,\mathbf{kk})\, dZ_1]$$

$$d\mathbf{Z}_w = \rho^{-1}\,(i\omega + \nu(k)\,\mathbf{kk})^{-1}\,[(\mathbf{B}\omega + \mathbf{C})\, dZ_\rho - \rho\, d\mathbf{Z}_2] \tag{2.3}$$

$$2b = \nu(k)\,\mathbf{kk} + \mathbf{Bk}, \qquad c = \mathbf{Ck}$$

Let us now put

$$\langle d\mathbf{Z}_2^* * d\mathbf{Z}_2\rangle = \boldsymbol{\varphi}(\mathbf{k})\, d\omega\, d\mathbf{k}, \qquad \boldsymbol{\varphi} = \boldsymbol{\psi} + i\boldsymbol{\chi}, \qquad \psi_{ij} = \psi_{ji} \tag{2.4}$$

$$\chi_{ij} = -\chi_{ji}, \qquad \langle dZ_1^*\, d\mathbf{Z}_2\rangle = \boldsymbol{\alpha}(\mathbf{k})\, d\omega\, d\mathbf{k}, \qquad \boldsymbol{\alpha} = \boldsymbol{\beta} + i\boldsymbol{\gamma}$$

$$\langle dZ_1^*\, dZ_1\rangle = \delta(\mathbf{k})\, d\omega\, d\mathbf{k}$$

Here ψ_{ij}, χ_{ij}, α_i, β_i and γ are some unknown functions of $\mathbf{k}$, and are independent of ω. We shall now compute the spectral densities

$$\langle dZ_\rho{}^* dZ_1\rangle = \rho\,[\omega^4 + (2b\omega + c)^2]^{-1}(\Psi_{\rho,1} + iX_{\rho,1})\,d\omega\,d\mathbf{k},$$
$$\Psi_{\rho,1} = (2b\omega + c)\,\mathbf{k}\beta + \omega^2\mathbf{k}\gamma + [\nu(k)\,\mathbf{kk}\,(2b\omega + c) + \omega^3]\,\delta$$
$$X_{\rho,1} = \omega^2\mathbf{k}\beta - (2b\omega + c)\,\mathbf{k}\gamma + [\nu(k)\,\mathbf{kk}\omega^2 - \omega\,(2b\omega + c)]\,\delta,$$
$$\langle dZ_\rho{}^* dZ_2\rangle = \rho\,[\omega^4 + (2b\omega + c)^2]^{-1}(\Psi_{\rho,2} + iX_{\rho,2})\,d\omega\,d\mathbf{k},$$
$$\Psi_{\rho,2} = (2b\omega + c)\,\mathbf{k}\psi - \omega^2\mathbf{k}\chi + [\nu(k)\,\mathbf{kk}\,(2b\omega + c) + \omega^3]\,\beta -$$
$$- [\nu(k)\,\mathbf{kk}\omega^2 - \omega\,(2b\omega + c)]\,\gamma, \quad \mathrm{X}_{\rho,2} = \omega^2\mathbf{k}\psi + (2b\omega + c)\,\mathbf{k}\chi +$$
$$+ [\nu(k)\,\mathbf{kk}\omega^2 - \omega\,(2b\omega + c)]\,\beta + [\nu(k)\,\mathbf{kk}\,(2b\omega + c) + \omega^3]\,\gamma \qquad (2.5)$$

Using the relations (2.5) and (2.3) together with the properties of the functions appearing in (2.4), we obtain the following expressions for the spectral densities of the observed random processes

$$\langle dZ_\rho{}^* dZ_\rho\rangle = \rho^2\,[\omega^4 + (2b\omega + c)^2]^{-1}\,L_{\rho,\rho}\,d\omega\,d\mathbf{k}$$
$$L_{\rho,\rho} = \mathbf{k}\psi\mathbf{k} + 2\mathbf{k}\,[(\nu(k)\,\mathbf{kk})\,\beta + \omega\gamma] + [\omega^2 + (\nu(k)\,\mathbf{kk})^2]\,\delta$$
$$\langle dZ_\rho{}^* d\mathbf{Z}_w\rangle = \rho\Phi^{-1}(\mathbf{L}_{\rho,w} + i\mathbf{M}_{\rho,w})\,d\omega\,d\mathbf{k} \qquad (2.6)$$
$$\mathbf{L}_{\rho,w} = (\nu(k)\,\mathbf{kk})\,[(\mathbf{B}\omega + \mathbf{C})\,L_{\rho,\rho} - \Psi_{\rho,2}] - \omega\mathrm{X}_{\rho,2}$$
$$\mathbf{M}_{\rho,w} = -\,\omega\,[(\mathbf{B}\omega + \mathbf{C})\,L_{\rho,\rho} - \Psi_{\rho,2}] - (\nu(k)\,\mathbf{kk})\,\mathrm{X}_{\rho,2}$$
$$\langle d\mathbf{Z}_w{}^* * d\mathbf{Z}_w\rangle = \Phi^{-1}(\mathbf{L}_{w,w} + i\mathbf{M}_{w,w})\,d\omega\,d\mathbf{k}$$
$$\mathbf{L}_{w,w} = [\omega^4 + (2b\omega + c)^2]\,\psi - [(\mathbf{B}\omega + \mathbf{C}) * \Psi_{\rho,2} + \Psi_{\rho,2} * (\mathbf{B}\omega + \mathbf{C})] +$$
$$+ (\mathbf{B}\omega + \mathbf{C}) * (\mathbf{B}\omega + \mathbf{C})\,L_{\rho,\rho}, \quad \mathbf{M}_{w,w} = [\omega^4 + (2b\omega + c)^2]\,\chi +$$
$$+ [\mathrm{X}_{\rho,2} * (\mathbf{B}\omega + \mathbf{C}) - (\mathbf{B}\omega + \mathbf{C}) * \mathrm{X}_{\rho,2}]$$
$$\Phi = [\omega^4 + (2b\omega + c)^2]\,[\omega^2 + (\nu(k)\,\mathbf{kk})^2]$$

Introducing now, analogously to (2.4), new unknown functions defining the average values of the products of $d\mathbf{Z}_3$ with other dZ_i, we can easily write expressions for the spectral densities of the correleations in which the random process $\mathbf{v}_L{}'$participates.

We should note, that above we have only considered a particular "natural" solution of the Eqs. (1.10) dependent on the presence of random magnitudes dZ_Q and dZ_F in the right-hand sides of these equations, and this corresponds to an investigation of the steady random motions in a two-phase system. In principle, we can use this method to describe both, the motions dependent on the usual turbulence in a mixture, and the specific pseudo-turbulent pulsations caused, in accordance with the physical model given in [1 – 3], by the force of gravity and the viscous interaction between the phases, both of them acting on the concentration fluctuations of a disperse system.

Since the transfer of impulse in the gaseous phase in neglected within the approximation considered, the turbulence of the supporting flow cannot influence the pulsations of the dispersed phase to any appreciable degree, and this leaves only the pseudoturbulent components of the latter (this corresponds to $F_L^{(g)} = 0$ in (1.7)).

The above may, of course, be no longer true when ρ are sufficiently small, since, if the Reynolds number of the averaged motion becomes sufficiently large, then the turbulent stresses in the gas will become comparable with the fluctuations of the forces between the phases and it will no longer be possible to neglect them.

On the other hand, when ρ is large, then the effect of the turbulence of the supporting flow on the random pulsations of the particles can be neglected, and this corresponds to the well known effect of "freezing out" the turbulence in the concentrated disperse system [7]. Another reason for treating the pseudoturbulence separately is, that, in many processes of considerable practical importance the Reynolds number of the average motion is not large and the usual type turbulence does not occur (pseudoliquefaction, pneumatic transport of granular materials under large loads e.a.).

From (2.6) we obtain the following relations for the pseudoturbulent pressure and viscosity of the dispersed phase:

$$\Pi(k) = \int\limits_{\omega}\int\limits_{k'>k} \mathbf{L}_{w,w}\frac{d\omega d\mathbf{k}'}{\omega}$$

$$\nu(k) = \alpha\int\limits_{\omega}\int\limits_{k'>k}\int\limits_{0}^{\infty}(\mathbf{L}_{w,w}\cos\omega\tau - \mathbf{M}_{w,w}\sin\omega\tau)\frac{d\omega d\mathbf{k}'}{\omega}d\tau \tag{2.7}$$

Thus the kinematic tensors introduced in Section 1 and describing the influence of the small scale pseudoturbulence on the large scale turbulence and on the averaged motion, can be written as functionals of the required functions of the wave vector.

To obtain these functions, we shall employ the equations for the simultaneous two-point correlations, which can be obtained in the usual manner from the equations of motion. However, we can easily see that the equations for various correlations of the processes $\mathbf{w}_L'$ and ρ_L' are sufficient in the present case, i.e. it is advisable to use Eqs. (2.1) themselves as a starting point in constructing the corresponding stochastic equations. Averaging over the periods of time which are large compared with the internal, and small compared with the internal time scale of the pseudoturbulence, leads to the disappearance of random quantities from the right-hand sides of the equations, and (2.1) then yields the stochastic equations of the form

$$\frac{\partial\rho_L'}{\partial t} + \rho\frac{\partial\mathbf{w}_L'}{\partial\mathbf{r}} = 0, \quad \rho\left(\frac{\partial}{\partial\mathbf{r}}\frac{\partial}{\partial\mathbf{r}}\right)\frac{\partial\mathbf{w}_L'}{\partial t} - \rho\left(\nu_L\frac{\partial}{\partial\mathbf{r}}\frac{\partial}{\partial\mathbf{r}}\right)\mathbf{w}_L' + \rho\left[-\left(\nu_L\frac{\partial}{\partial\mathbf{r}}\right)\times\right.$$
$$\left.\times\left(\frac{\partial}{\partial\mathbf{r}}\frac{\partial}{\partial\mathbf{r}}\right) + \frac{\beta K}{(1-\rho)^2}\frac{\partial}{\partial\mathbf{r}}\right]\frac{\partial\mathbf{w}_L'}{\partial\mathbf{r}} + \left(\frac{\partial\mathbf{w}}{\partial t} - \mathbf{g}\right)\left(\frac{\partial}{\partial\mathbf{r}}\frac{\partial}{\partial\mathbf{r}}\right)\rho_L' - \left(\mathbf{A}\frac{\partial}{\partial\mathbf{r}}\right)\frac{\partial\rho_L'}{\partial\mathbf{r}} = 0 \tag{2.8}$$

The computations for the simultaneous two-point correlations, yield subsequently

$$\frac{\partial Q^{(L)}_{\rho,\rho}}{\partial t} + \frac{\partial}{\partial\boldsymbol{\xi}}(\mathbf{Q}^{(L)}_{\rho,w} - \mathbf{Q}^{(L)}_{w,\rho}) = 0, \qquad \boldsymbol{\xi} = \mathbf{r}_B - \mathbf{r}_A$$
$$\rho\left(\frac{\partial}{\partial\boldsymbol{\xi}}\frac{\partial}{\partial\boldsymbol{\xi}}\right)\left[\frac{\partial}{\partial t} - \left(\nu_L\frac{\partial}{\partial\boldsymbol{\xi}}\frac{\partial}{\partial\boldsymbol{\xi}}\right)\right]\mathbf{Q}^{(L)}_{\rho,w} + \rho\left[-\left(\nu_L\frac{\partial}{\partial\boldsymbol{\xi}}\right)\left(\frac{\partial}{\partial\boldsymbol{\xi}}\frac{\partial}{\partial\boldsymbol{\xi}}\right) + \right.$$
$$\left.+\frac{\beta k}{(1-\rho)^2}\frac{\partial}{\partial\boldsymbol{\xi}}\right]\frac{\partial\mathbf{Q}^{(L)}_{\rho,w}}{\partial\boldsymbol{\xi}} + \left(\frac{\partial\mathbf{w}}{\partial t} - \mathbf{g}\right)\left(\frac{\partial}{\partial\boldsymbol{\xi}}\frac{\partial}{\partial\boldsymbol{\xi}}\right)Q^{(L)}_{\rho,\rho} - \left(\mathbf{A}\frac{\partial}{\partial\boldsymbol{\xi}}\right)\frac{\partial Q^{(L)}_{\rho,\rho}}{\partial\boldsymbol{\xi}} -$$
$$-\rho^2\left(\frac{\partial}{\partial\boldsymbol{\xi}}\frac{\partial}{\partial\boldsymbol{\xi}}\right)\frac{\partial\mathbf{Q}^{(L)}_{w,w}}{\partial\boldsymbol{\xi}}, \quad \rho\left(\frac{\partial}{\partial\boldsymbol{\xi}}\frac{\partial}{\partial\boldsymbol{\xi}}\right)\left[\frac{\partial}{\partial t} - 2\left(\nu_L\frac{\partial}{\partial\boldsymbol{\xi}}\frac{\partial}{\partial\boldsymbol{\xi}}\right)\right]\mathbf{Q}^{(L)}_{w,w} + \tag{2.9}$$
$$+\rho S\left\{\left[-\left(\nu_L\frac{\partial}{\partial\boldsymbol{\xi}}\right)\left(\frac{\partial}{\partial\boldsymbol{\xi}}\frac{\partial}{\partial\boldsymbol{\xi}}\right) + \frac{\beta K}{(1-\rho)^2}\frac{\partial}{\partial\boldsymbol{\xi}}\right] * \frac{\partial\mathbf{Q}^{(L)}_{w,w}}{\partial\boldsymbol{\xi}}\right\} + S\left\{\left[\left(\frac{\partial\mathbf{w}}{\partial t} - \right.\right.\right.$$
$$\left.\left.\left.-\mathbf{g}\right)\left(\frac{\partial}{\partial\boldsymbol{\xi}}\frac{\partial}{\partial\boldsymbol{\xi}}\right) - \left(\mathbf{A}\frac{\partial}{\partial\boldsymbol{\xi}}\right)\frac{\partial}{\partial\boldsymbol{\xi}}\right] * \mathbf{Q}^{(L)}_{\rho,w}\right\}; \qquad \frac{\partial\mathbf{Q}^{(L)}_{w,w}}{\partial\boldsymbol{\xi}} = \frac{\partial\mathbf{Q}^{(L)}_{wi,w}}{\partial\xi_i}$$
$$Q^{(L)}_{\psi,\varphi}(t,\ \boldsymbol{\xi}) = \langle\psi_L'(t,\ \mathbf{r}_A)\,\varphi_L'(t,\ \mathbf{r}_B)\rangle, \quad (S\{\mathbf{a}\})_{ij} = a_{ij} + \dot{a}_{ji}$$

where A and B denote two points in space. Eqs. (2.9) represent the generalization of the well known correlation equations describing the turbulence of a single phase fluid.

Corresponding energy spectrum equations (i.e. Fourier transforms of (2.9)) have the form

$$\int_0^k \left\{ \frac{\partial E_{\rho,\rho}}{\partial t} + i\mathbf{k}\left(\mathbf{E}_{\rho,w} - \mathbf{E}_{w,\rho}\right) \right\} k^2 dk = 0$$

$$\int_0^k \left\{ \rho \left[\frac{\partial}{\partial t} + (\boldsymbol{\nu}(k)\,\mathbf{kk}) \right] \mathbf{E}_{\rho,w} + \rho \mathbf{B}(\mathbf{k}\mathbf{E}_{\rho,w}) - \mathbf{C}E_{\rho,\rho} - \rho^2 \mathbf{k}\mathbf{E}_{w,w} \right\} k^2 dk = 0$$

$$\int_0^k \left\{ \rho \left[\frac{\partial}{\partial t} + 2(\boldsymbol{\nu}(k)\,\mathbf{kk}) \right] \mathbf{E}_{w,w} + \rho(\mathbf{B} * \mathbf{k}\mathbf{E}_{w,w} + \mathbf{E}_{w,w}\mathbf{k} * \mathbf{B}) - \right.$$
$$\left. - (\mathbf{C} * \mathbf{E}_{\rho,w} + \mathbf{E}_{w,\rho} * \mathbf{C}) \right\} k^2 dk = 0 \qquad (2.10)$$

Here functions E represent the spectral densities of simultaneous two-point correlations, and are equal to the spectral densities of the corresponding nonsimultaneous two-point correlations integrated along the frequency axis.

The assumption that the mean flow is much larger than the pseudoturbulent flow implies, that we only need to consider the steady state problem. Using Expressions (2.6) we obtain the spectral equations in the form

$$\mathbf{k} \int_{-\infty}^{\infty} \int_0^k \mathbf{M}_{\rho,w} \frac{d\omega k^2 dk}{\Phi} = 0, \quad \int_{-\infty}^{\infty} \int_0^k \left\{ \boldsymbol{\nu}(k)\,\mathbf{kk}\, \frac{\mathbf{L}_{\rho,w} + i\mathbf{M}_{\rho,w}}{\Phi} + \right.$$
$$+ \mathbf{B}\left(\mathbf{k}\frac{\mathbf{L}_{\rho,w} + i\mathbf{M}_{\rho,w}}{\Phi}\right) - \mathbf{C}\,\frac{\omega^2 + (\boldsymbol{\nu}(k)\,\mathbf{k}\mathbf{k})^2}{\Phi} L_{\rho,\rho} -$$
$$\left. - \mathbf{k}\,\frac{\mathbf{L}_{w,w} + i\mathbf{M}_{w,w}}{\Phi} \right\} d\omega\, k^2 dk = 0, \quad \int_{-\infty}^{\infty} \int_0^k \left\{ 2(\boldsymbol{\nu}(k)\,\mathbf{kk})\, \frac{\mathbf{L}_{w,w} + i\mathbf{M}_{w,w}}{\Phi} + \right.$$
$$+ \mathbf{B} * \mathbf{k}\, \frac{\mathbf{L}_{w,w} + i\mathbf{M}_{w,w}}{\Phi} + \mathbf{k}\, \frac{\mathbf{L}_{w,w} - i\mathbf{M}_{w,w}}{\Phi} * \mathbf{B} -$$
$$\left. - \mathbf{C} * \frac{L_{\rho,w} + i\mathbf{M}_{\rho,w}}{\Phi} - \frac{\mathbf{L}_{\rho,w} - i\mathbf{M}_{\rho,w}}{\Phi} * \mathbf{C} \right\} d\omega\, k^2 dk = 0 \qquad (2.11)$$

where $\boldsymbol{\nu}(k)$ is the tensor functional (2.7) of the required functions. When the integration with respect to k is performed in (2.10) and (2.11), the argument k of this tensor is taken as a parameter.

First Eq. of (2.11) is scalar and real, the second one is equivalent to six, while the third is equivalent to nine real scalar equations. Thus we have sixteen nonlinear integro-differential equations for defining sixteen unknowns in (2.4). These, obviously, should become zero as $k \to 0$ and $k \to \infty$. In addition, we have a normalizing condition according to which the expression for the mean square fluctuation of the concentration of the system over large volumes should coincide with that, obtained from the statistical analysis (see [2] and [8]). It can be shown that the points $k = 0$ and $k = \infty$ represent the branch points of the solutions of (2.11), therefore the normalizing condition does not violate the definiteness of the system.

Expression (2.11) shows clearly the analogy with the Heisenberg [5] hypothesis concerning the spectral energy transfer. Indeed, (2.11) corresponds to a model, according to which the influence of the vortices with wave number greater than k on the perturbations with the wave number equal to k resulting in the net loss of energy by the latter is such, as would be caused by some pseudoturbulent viscisity resulting from the presence of these vortices.

When dealing, for example, with problems of diffusion of some extra component in the disperse system, we might become concerned with pseudoturbulent gas pulsations. In this case it will be necessary to introduce new unknown functions of k defining the mean values $\langle dZ_3{}^* * dZ_3 \rangle$, $\langle dZ_3{}^* * dZ_2 \rangle$ and $\langle dZ_3{}^* dZ_1 \rangle$. There are twenty four such functions and the first Eq. of (2.2) defining the relative velocity of gas yields easily, using the method given above, the same number of new integro-differential spectral equations defining these functions.

Thus we see that the most important simplification connected with neglecting the transfer of impulse in the gaseous phase consists of the fact, that it makes it possible to isolate a closed system of spectral equations for the quantities characterizing only the dispersed phase and, consequently, to reduce considerably the number of equations. However even in this simplest case, solution of the system of spectral equations is very difficult and the difficulties begin already during the final stages of formulating the problem, i.e. during the integration with respect to ω in (2.11). Therefore, further attempts at constructing simpler models of pseudoturbulence in a disperse system based on simplification of (2.11), may be of interest.

We note that in very concentrated disperse systems approaching the condition of dense packing, the effects due to mechanical interaction between the suspended particles, become significant. This, however, does not present an insurmaountable difficulty — we can overcome it in the first approximation by replacing α in (2.7) with some function of ρ, which would become infinite as $\rho \to \rho_*$, where ρ_* denotes the concentration of the system in the state of close packing. The form of this function follows from the analysis of the transport processes in dense gases [1 and 3].

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